Lecture 34: Nov 20

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations
- Conditional Distributions
- Independence

Continuous Bivariate RVs The random variables X and Y are said to be *jointly continuous* if there exists a function $f_{X,Y}(x, y)$, such that for any Borel set B of 2-tuples in $\mathbb{R}^2$,

$$\Pr\{(X, Y) \in B\} = \int\int_{(x,y)\in B} f_{X,Y}(x, y)dxdy.$$

The function $f_{X,Y}(x, y)$ is called the *joint probability density function* for X and Y. It follows in this case that

$$F_{X,Y}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s, t)dtds,$$
$$f_{X,Y}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

Properties of the bivariate pdf

- $f_{X,Y}(x, y) \geqslant 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$
- $f_{X,Y}(x, y)$ is not a probability, but can be thought of as a relative probability of (X, Y) falling into a small rectangle located at (x, y):

$$\Pr\{x < X \leqslant x + dx, y < Y \leqslant y + dy\} \approx f(x, y)dxdy$$

- The *marginal probability density functions* for X and Y can be obtained as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Example 1

$$F_{X,Y}(x,y) = xy \qquad 0 < x \leqslant 1, 0 < y \leqslant 1$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Example 2

$$F_{X,Y}(x,y) = x - x\log\frac{x}{y} \qquad 0 < x \leqslant y \leqslant 1$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Note: Once we have $f_X(x)$ and $f_Y(y)$, we can obtain $F_X(x)$ and $F_Y(y)$ directly. Double check: $F_X(x) = F_{X,Y}(x,\infty)$.

Conditional Distributions

Conditional Distributions - Discrete Recall if A and B are two events, the probability of A conditional on B is:

$$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$$

Defining the events $A = \{Y = y\}$ and $B = \{X = x\}$, it follows that

$$\begin{aligned}\Pr\{Y = y|X = x\} &= \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \\ &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= f_{Y|X}(y|x)\end{aligned}$$

This is called the *conditional probability mass function* of Y given X.

Example: Discrete Back to the fair coin example. From the joint pmf of X and Y, we can derive all the conditional pmfs:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

Conditional Distribution - Continuous If $F(x,y)$ is absolutely continuous, we define the conditional density of X given Y as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

Example 1

$$\begin{array}{lll}
F_{XY}(x,y) = xy & 0 < x < 1, & 0 < y < 1 \\
f_{XY}(x,y) = 1 & 0 < x < 1, & 0 < y < 1 \\
f_X(x) = 1 & 0 < x < 1 & \\
f_Y(y) = 1 & 0 < y < 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1 & 0 < x < 1 & (0 < y < 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 1 & 0 < y < 1 & (0 < x < 1)
\end{array}$$

Note: Here we get that the conditional densities are the same as the marginals. This means X and Y are independent.

Example 2

$$\begin{array}{lll}
F_{XY}(x,y) = x - x\log\frac{x}{y} & 0 < x \leqslant y \leqslant 1 & \\
f_{XY}(x,y) = 1/y & 0 < x \leqslant y \leqslant 1 & \\
f_X(x) = -\log x & 0 < x \leqslant 1 & \\
f_Y(y) = 1 & 0 < y \leqslant 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1/y & 0 < x \leqslant y & (0 < y \leqslant 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = -\frac{1}{y\log x} & x \leqslant y \leqslant 1 & (0 < x \leqslant 1)
\end{array}$$

- Y is marginally uniform, but not conditionally uniform.
- X is conditionally uniform, but not marginally uniform.

Independent Random Variables

Independence The random variable X and Y are said to be *independent* if for any two Borel sets A and B,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B)$$

All events defined in terms of X are independent of all events defined in terms of Y.

Using the Kolmogorov axioms of probability, it can be shown that X and Y are independent if and only if $\forall(x, y)$ (except possibly for sets of probability 0)

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

or in terms of pmfs (discrete) and pdfs (continuous)

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Checking independence

- A necessary condition for independence of X and Y is that their joint pdf/pmf has positive probability on a rectangular domain.
- If the domain is rectangular, one can try to write the joint pdf/pmf as a product of functions of x and y only.

Example Two points are selected randomly on a line of length a so as to be on opposite sides of the mid-point of the line. Find the probability that the distance between them is less than $a/3$.

Solution:
Let X be the coordinate of a point selected randomly in $[0, a/2]$ and Y be the coordinate of a point selected randomly in $[a/2, a]$. Assume X and Y are independent and uniform over its interval. The joint density is

$$f_{X,Y}(x, y) = 4/a^2, \quad 0 \leqslant x \leqslant a/2, a/2 \leqslant y \leqslant a$$

Therefore, the solution is

$$\Pr(Y - X < a/3) =$$

Example: Buffon's Needle A table is ruled with lines distance 1 unit apart. A needle of length $L \leqslant 1$ is thrown randomly on the table. What is the probability that the needle intersects a line?
Solution:
Define two random variables:

- X: distance from low end of the needle to the nearest line above
- θ: angle from the vertical to the needle.

By "random", we assume X and θ are independent, and

$$X \sim U(0, 1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\theta}(x, \theta) = 1/\pi, \quad 0 \leqslant x \leqslant 1, -\pi/2 \leqslant \theta \leqslant \pi/2$$

For the needle to intersect a line, we need $X < L\cos(\theta)$.