Math 3070/6070 Homework 4
Due: Oct 21st, 2024

1. (2.6) In each of the following find the pdf of Y and show that the pdf integrates to 1.
   1. $f_X(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$; $Y = |X|^3$
   2. $f_X(x) = \frac{3}{8}(x+1)^2, -1 < x < 1$; $Y = 1 - X^2$
   3. $f_X(x) = \frac{3}{8}(x+1)^2, -1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$
2. (2.9) If the random variable X has pdf

$$f(x) = \begin{cases} \frac{x-1}{2}, & 1 < x < 3 \\ 0, & otherwise, \end{cases}$$

   find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a $uniform(0,1)$ distribution.
3. (2.13) Consider a sequence of independent coin flips, each of which has probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. (For example, $X = 3$ if either TTTH or HHHT is observed.) Find the distribution of X and find EX.
4. (2.15) Suppose the pdf $f_X(x)$ of a random variable X is an *even* function. ($f_X(x)$ is an even function if $f_X(x) = f_X(-x)$ for every x.) Show that
   1. X and $-X$ are identically distributed.
   2. $M_X(t)$ is symmetric about 0.
5. (2.33) In each of the following cases verify the expression given for the moment generating function, and in each case use the mgf to calculate EX and $Var(X)$.
   1. $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $M_X(t) = e^{\lambda(e^t - 1)}$, $x = 0, 1, \ldots; \lambda > 0$.
   2. $\Pr(X = x) = p(1-p)^x$, $M_X(t) = \frac{p}{1-(1-p)e^t}$, $x = 0, 1, \ldots; 0 < p < 1$.
   3. $f_X(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}$, $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$, $-\infty < x < \infty; -\infty < \mu < \infty, \sigma > 0$.
6. (3.3) The flow of traffic at certain street corners can sometimes be modeled as a sequence of Bernoulli trials by assuming that the probability of a car passing during any given second is a constant p and that there is no interaction between the passing of cars at different seconds. If we treat seconds as indivisible time units (trials), the Bernoulli model applies. Suppose a pedestrian can cross the street only if no car is to pass during the next 3 seconds. Find the probability that the pedestrian has to wait for exactly 4 seconds before starting to cross.
7. (3.4) A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean number of trials if
   1. unsuccessful keys are not eliminated from further selections.
   2. unsuccessful keys are eliminated.

8. (3.12) Suppose X has a *Binomial*(n, p) distribution and let Y have a negative binomial(r, p) distribution. Show that $F_X(r-1) = 1 - F_Y(n-r)$.

9. (3.13) A *truncated* discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range $0, 1, 2, \ldots$ and the 0 cannot be observed (as is usually the case), the $0 -$ *truncated* random variable X_T has pmf

$$\Pr(X_T = x) = \frac{\Pr(X = x)}{\Pr(X > 0)}, x = 1, 2, \ldots.$$

   Find the pmf, mean and variance of $0-$truncated random variable starting from

   1. $X \sim$ Poisson(λ).
   2. $X \sim$ negative binomial(r, p), as in (3.2.10) of the textbook.